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Note

A note on domination, girth and minimum degree

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Abstract

Let G be a graph of order n, minimum degree $\delta \geqslant 2$, girth $g \geqslant 5$ and domination number γ. In 1990 Brigham and Dutton [Bounds on the domination number of a graph, Q. J. Math., Oxf. II. Ser. 41 (1990) 269–275] proved that $\gamma \leqslant \lceil n/2 - g/6 \rceil$. This result was recently improved by Volkmann [Upper bounds on the domination number of a graph in terms of diameter and girth, J. Combin. Math. Combin. Comput. 52 (2005) 131–141; An upper bound for the domination number of a graph in terms of order and girth, J. Combin. Math. Combin. Comput. 54 (2005) 195–212] who for $i \in \{1, 2\}$ determined a finite set of graphs $\mathscr{G}_i$ such that $\gamma \leqslant \lceil n/2 - g/6 - (3i+3)/6 \rceil$ unless G is a cycle or $G \in \mathscr{G}_i$.

Our main result is that for every $i \in \mathbb{N}$ there is a finite set of graphs $\mathscr{G}_i$ such that $\gamma \leqslant n/2 - g/6 - i$ unless G is a cycle or $G \in \mathscr{G}_i$. Furthermore, we conjecture another improvement of Brigham and Dutton's bound and prove a weakened version of this conjecture.

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Keywords: Domination; Girth; Minimum degree

1. Introduction

We consider finite and simple graphs $G = (V_G, E_G)$ with vertex set V_G and edge set E_G. The order of G is denoted by $n(G) = |V_G|$. The degree and the neighbourhood of a vertex $u \in V_G$ in the graph G are denoted by $d_G(u)$ and $N_G(u)$, respectively. The minimum degree of G is denoted by $\delta(G)$ and the girth of G—which is the minimum length of a cycle of G—is denoted by $g(G)$. The domination number $\gamma(G)$ of G is the minimum cardinality of a set $D \subseteq V_G$ with $N_G(u) \cap D \neq \emptyset$ for all $u \in V_G \backslash D$. The subgraph of G that is induced by a set $U \subseteq V_G$ is denoted by $G[U]$.

In 1990 Brigham and Dutton [1] observed that the deletion of a shortest cycle C from a graph G with minimum degree at least 2 and girth at least 5 leaves a graph H without isolated vertices. Together with Ore's bound [4]

$$\gamma \leqslant \frac{n}{2} \tag{1}$$

on the domination number γ of graphs of order n without isolated vertices this immediately implies a bound of the form

$$\gamma(G) \leqslant \gamma(H) + \gamma(C) \leqslant \left\lceil \frac{n(G)}{2} - \frac{g(G)}{6} \right\rceil \tag{2}$$

for such graphs (cf. [2]).

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In 1989 McCuaig and Shepherd [3] proved that

$$\gamma(G) \leqslant \frac{2n(G)}{5} \tag{3}$$

for all graphs G of minimum degree at least 2 that do not belong to a finite set $\mathscr{B}$ of exceptional graphs. In view of (3) it should be possible to considerably improve the crude argument that led to (2). In fact, we believe that there should be a bound on the domination number of graphs G of minimum degree at least 2 that is always bounded by $2n(G)/5 + \mathrm{O}(1)$ and tends to $n(G)/3 + \mathrm{O}(1)$ as $n(G)/g(G)$ tends to 1. To incite related research we pose the following conjecture.

Conjecture 1. If G is a graph of order n, minimum degree $\delta \geqslant 2$, girth $g \geqslant 5$ and domination number γ, then

$$\gamma \leqslant \frac{n}{3} + \frac{2n}{3(g-1)}.$$

In the present paper we pursue an improvement of (2) in a form that was recently considered by Volkmann. In [5,6] Volkmann determined two finite sets $\mathscr{G}_1$ and $\mathscr{G}_2$ of graphs such that for $i = 1, 2$ and every graph G of minimum degree at least 2 and girth at least 5 that is not a cycle

$$\gamma(G) \leqslant \left\lceil \frac{n(G)}{2} - \frac{g(G)}{6} - \frac{(3i+3)}{6} \right\rceil \tag{4}$$

unless $G \in \mathscr{G}_i$. Our main result is that such a result holds for every $i \in \mathbb{N}$. Furthermore, we prove a weakened form of Conjecture 1.

2. Results

We immediately proceed to our main result.

Theorem 1. *Let $k \in \mathbb{N}$. There is a finite set $\mathscr{G}_k$ of graph such that if G is a graph of order n, minimum degree $\delta \geqslant 2$, girth $g \geqslant 5$ and domination number γ that is not a cycle and does not belong to $\mathscr{G}_k$, then*

$$\gamma \leqslant \frac{n}{2} - \frac{g}{6} - k.$$

Proof. Let G be a graph of order n, minimum degree $\delta \geqslant 2$, girth $g \geqslant 5$ and domination number γ. We consider different cases.

Case 1: $n \geqslant \frac{5}{3}g + 10k$ or $g < \max\{12k + 18, 6k + 53\}$.

Without loss of generality, we assume that $\mathscr{G}_k$ contains all graph in $\mathscr{B}$ from (3) and also all graphs of order less than $\frac{5}{3}\max\{12k + 18, 6k + 53\} + 10k$.

If $n \geqslant \frac{5}{3}g + 10k$, then either $G \in \mathscr{G}_k$ or, by (3), $\gamma \leqslant 2n/5 \leqslant n/2 - g/6 - k$ and the result holds. If $n < \frac{5}{3}g + 10k$, then $G \in \mathscr{G}_k$ and the result holds. Hence we may assume $n < \frac{5}{3}g + 10k$ and $g \geqslant \max\{12k + 18, 6k + 53\}$.

Case 2: G has a cycle of length $l \geqslant g + 6k + 4$.

Let C be a shortest such cycle of G.

First, we assume that $N_G(u) \nsubseteq V_C$ for all $u \in V_G \setminus V_C$. This implies that the graph $H = G[V_G \setminus V_C]$ has minimum degree at least 1. Hence, by (1),

$$\gamma \leqslant \gamma(H) + \gamma(C) \leqslant \frac{n-l}{2} + \frac{l+2}{3} \leqslant \frac{n}{2} - \frac{g}{6} - k$$

and the result holds.

Thus we may assume that there is a vertex $u \in V_G \setminus V_C$ with $|N_G(u) \cap V_C| \geqslant 2$. Let $v, w \in N_G(u) \cap V_C$. The path vuw together with the two paths in C from v to w form two cycles. In view of Case 1, the length of these two paths is at least 3. Considering the cycle formed with the shorter of the two paths, we obtain $g \leqslant l/2 + 2$ which implies $l \geqslant 2g - 4$.

If $l \geqslant 2g+12k+4$, then vuw together with the longer of the two path forms a cycle of length at least $l/2+2 \geqslant g+6k+4$ that is shorter than C. Since this contradicts the choice of C, we have

$$2g-4 \leqslant l \leqslant 2g+12k+3.$$

If there is a vertex $u \in V_G \setminus V_C$ with $|N_G(u) \cap V_C| \geqslant 3$, then G contains a cycle of length at most $l/3+2$ which implies the contradiction $l \geqslant 3g-6 > 2g+12k+3$. Hence every vertex in $V_G \setminus V_C$ has at most two neighbours in V_C.

Let

$$C : x_1 x_2 x_3 \dots x_l x_1$$

and let

$$V_1 = \{x_i \mid 1 \leqslant i \leqslant g-4\},$$

$$V_2 = \{x_i \mid g-3 \leqslant i \leqslant 2g-8\} \text{ and}$$

$$V_3 = \{x_i \mid 2g-7 \leqslant i \leqslant l\}.$$

Note that $l \leqslant 2g+12k+3 \leqslant (2g-7)+(g-5) = 3g-12$. If there is a vertex $u \in V_G \setminus V_C$ with $|N_G(u) \cap V_i| \geqslant 2$ for some $1 \leqslant i \leqslant 3$, then G contains a cycle of length at most $2+(g-5)$ which implies the contradiction. Hence no such vertex exists.

For $1 \leqslant i < j \leqslant 3$ let $V_{i,j}$ denote the set of vertices $u \in V_G \setminus V_C$ with $d_G(u) = 2$ $N_G(u) \cap V_i \neq \emptyset$ and $N_G(u) \cap V_j \neq \emptyset$. Let $n' = |V_{1,2} \cup V_{1,3} \cup V_{2,3}|$.

If $n' \geqslant 16$, then we may assume without loss of generality that $|V_{1,2}| \geqslant 6$. By Ramsey's theorem ($r(3,3)=6$), there are three vertices $u_1, u_2, u_3 \in V_{1,2}$ with $N_G(u_i) \cap V_1 = \{v_i\}$ and $N_G(u_i) \cap V_2 = \{w_i\}$ for $i = 1, 2, 3$ such that the vertices in

$$N' = N_G(u_1) \cup N_G(u_2) \cup N_G(u_3)$$

appear on C either in the order

$$v_1, v_2, v_3, w_1, w_2, w_3$$

or in the order

$$v_1, v_2, v_3, w_3, w_2, w_1.$$

In both cases (cf. Fig. 1) there are two cycles consisting of edge-disjoint paths on C whose vertices lie totally within $V_1 \cup V_2$ and the paths $v_i u_i w_i$ for $i = 1, 2, 3$. Adding the length of these two cycles yields at most $((g-5)+2+2)+((g-5)+2+2) = 2g-2$ which implies the contradiction that one of the two cycles has length less than G. Therefore, $n' \leqslant 15$.

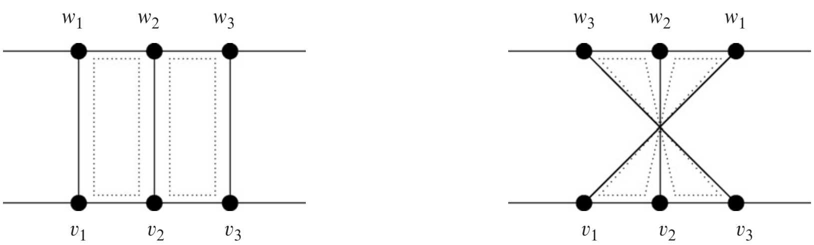

Fig. 1.

Since the graph $H = G[V_G \setminus (V_{1,2} \cup V_{1,3} \cup V_{2,3} \cup V_C)]$ has minimum degree at least 1, we obtain, by (1),

$$\begin{aligned}
\gamma &\leqslant \gamma(H) + |V_{1,2} \cup V_{1,3} \cup V_{2,3}| + \gamma(C) \\
&\leqslant \frac{n - n' - l}{2} + n' + \frac{l+2}{3} \\
&\leqslant \frac{n}{2} - \frac{l}{6} + \frac{n'}{2} + \frac{2}{3} \\
&\leqslant \frac{n}{2} - \frac{l}{6} + \frac{15}{2} + \frac{2}{3} \\
&\leqslant \frac{n}{2} - \frac{2g-4}{6} + \frac{15}{2} + \frac{2}{3} \\
&\leqslant \frac{n}{2} - \frac{g}{6} - k
\end{aligned}$$

and the result holds. This implies that we may assume that G has no cycle of length at least $g + 6k + 4$.

Case 3: Not all cycles of G are edge-disjoint.

This easily implies the existence of two vertices $u, v \in V_G$ such that there are three internally vertex-disjoint paths P, Q and R from u to v. Let P, Q and R be of lengths l_1, l_2 and l_3. By Case 2, $g \leqslant l_1 + l_2, l_2 + l_3, l_1 + l_3 \leqslant g + 6k + 4$. Let $U = V_P \cup V_Q \cup V_R$. By Cases 1 and 2, no vertex in $V_G \setminus U$ has two neighbours in u. Therefore, the graph $H = G[V_G \setminus U]$ has minimum degree at least 1. It is easy to see that $\gamma(G[U]) \leqslant (l_1 + l_2 + l_3)/3 + 1$. Since $l_1 + l_2, l_2 + l_3, l_1 + l_3 \geqslant g$, we have $l_1 + l_2 + l_3 \geqslant \frac{3}{2}g$. By (1), we obtain

$$\begin{aligned}
\gamma &\leqslant \gamma(H) + \gamma(G[U]) \\
&\leqslant \frac{n - (l_1 + l_2 + l_3 - 1)}{2} + \frac{l_1 + l_2 + l_3}{3} + 1 \\
&\leqslant \frac{n}{2} - \frac{l_1 + l_2 + l_3}{6} + \frac{3}{2} \\
&\leqslant \frac{n}{2} - \frac{g}{4} + \frac{3}{2} \\
&\leqslant \frac{n}{2} - \frac{g}{6} - k.
\end{aligned}$$

Hence, we may assume that all cycles of G are edge-disjoint.

Since G is not a cycle, let C_1 and C_2 be two cycles of length l_1 and l_2 in G. By Case 3, no vertex in $V_G \setminus (V_{C_1} \cup V_{C_2})$ has two neighbours in either V_{C_1} or V_{C_2}. By Cases 1 and 3, there is at most one vertex that has a neighbour both in V_{C_1} and V_{C_2}.

If the graph $H = G[V_G \setminus (V_{C_1} \cup V_{C_2})]$ has minimum degree at least 1, then, by (1), we obtain

$$\begin{aligned}
\gamma &\leqslant \gamma(H) + \gamma(G[V_{C_1} \cup V_{C_2}]) \\
&\leqslant \frac{n - (l_1 + l_2 - 1)}{2} + \frac{l_1 + 2}{3} + \frac{l_2 + 2}{3} \\
&\leqslant \frac{n}{2} - \frac{l_1}{6} - \frac{l_2}{6} + \frac{1}{2} + \frac{2}{3} + \frac{2}{3} \\
&\leqslant \frac{n}{2} - \frac{g}{3} + \frac{1}{2} + \frac{2}{3} + \frac{2}{3} \\
&\leqslant \frac{n}{2} - \frac{g}{6} - k.
\end{aligned}$$

Hence, we may assume that the graph H has an isolated vertex. This implies that there is a unique $u \in V_G \setminus (V_{C_1} \cup V_{C_2})$ with $d_G(u) = 2$, $N_G(u) \cap V_{C_1} \neq \emptyset$ and $N_G(u) \cap V_{C_2} \neq \emptyset$. Now the graph $H' = G[V_G \setminus (\{u\} \cup V_{C_1} \cup V_{C_2})]$ has minimum degree at least 1, and a similar argument as above implies the desired result. This completes the proof. □

We close with a weakened form of Conjecture 1.

Proposition 1. *If G is a graph of order n, minimum degree $\delta \geqslant 2$, girth $g \geqslant 5$ and domination number γ such that $\mathrm{dist}_G(u, v) \geqslant g$ for all $u, v \in V_G$ with $d_G(u), d_G(v) \geqslant 3$, then*

$$\gamma \leqslant \frac{n}{3} + \frac{2n}{3(g-1)}.$$

Proof. Obviously, we may assume that G is connected. We prove the result by induction on the order of G.

Clearly, $n \geqslant g$. If $n = g$, then G is a cycle and the result is obvious. Hence we may assume that $n > g$ and that G is not a cycle.

If there is a path $u_0 u_1 u_2 \ldots u_l$ in G with $d_G(u_0), d_G(u_l) \geqslant 3$ and $d_G(u_i) = 2$ for $1 \leqslant i \leqslant l-1$, then $l \geqslant g$. Since the graph $H = G[V_G \setminus \{u_1, u_2, \ldots, u_{l-1}\}]$ satisfies the assumptions of the theorem and has smaller order than G, we have, by induction,

$$\begin{aligned}
\gamma &\leqslant \gamma(H) + \gamma(G[\{u_1, u_2, \ldots, u_{l-1}\}]) \\
&\leqslant \frac{n-(l-1)}{3} + \frac{2(n-(l-1))}{3(g-1)} + \frac{(l-1)+2}{3} \\
&\leqslant \frac{n}{3} + \frac{2n}{3(g-1)} - \frac{2(l-1)}{3(g-1)} + \frac{2}{3} \\
&\leqslant \frac{n}{3} + \frac{2n}{3(g-1)}.
\end{aligned}$$

Hence we may assume that no such path exists. This implies that there is a unique vertex v_0 of degree at least 3. Note that $d_G(v_0)$ is necessarily even in this case which implies $d_G(v_0) \geqslant 4$. Let $v_0 v_1 v_2 \ldots v_{l-1} v_0$ be a cycle in G with $d_G(v_i) = 2$ for $1 \leqslant i \leqslant l-1$. Clearly, $l \geqslant g$ and a similar argument as above implies the desired result. This completes the proof. □

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